

# Precision test of AdS/CFT in Lunin-Maldacena background

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**ABSTRACT:** We obtain the solutions and explicitly calculate the energy for a class of two-spin semiclassical string states in the Lunin-Maldacena background. These configurations are the  $\beta$ -deformed versions of the folded string solutions in  $AdS_5 \times S^5$  background. They correspond to certain single trace operators in the  $\mathcal{N} = 1$  superconformal  $\beta$  deformation of  $\mathcal{N} = 4$  Yang-Mills. We calculate the one loop anomalous dimension for the dual single trace operators from the associated twisted spin chain with a general two-cut distribution of Bethe roots. Our results show a striking match between the two calculations. We demonstrate the natural identification of parameters on the two sides of the analysis, and explain the significance of the Virasoro constraint associated with the winding motion of semiclassical strings from the perspective of the spin chain solution.

**KEYWORDS:** Gauge-gravity correspondence, AdS-CFT Correspondence.

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## 1. Introduction

The AdS/CFT correspondence [1–4] in its original form states the equivalence between Type IIB superstring theory propagating on  $AdS_5 \times S^5$  and  $\mathcal{N} = 4$ ,  $SU(N)$  supersymmetric Yang-Mills (SYM) theory in the large  $N$  limit, defined on the boundary of  $AdS_5$ . Extending this duality to situations with lower supersymmetry (SUSY) is most naturally and directly achieved by considering deformations of  $\mathcal{N} = 4$  theory which translate to, via the precise dictionary of the original correspondence, specific deformations of the  $AdS_5 \times S^5$  background. A particularly interesting class of such deformations are the exactly marginal ones preserving  $\mathcal{N} = 1$  SUSY, elucidated by Leigh and Strassler in [5] and mentioned by earlier authors [6]. One of these deformations, the so-called  $\beta$  deformation, and its large  $N$  gravity dual constructed by Lunin and Maldacena [7] will be the focus of this paper.

In particular, we will look at certain semiclassical string solutions in the Lunin-Maldacena background, which are the  $\beta$ -deformed versions of the two-spin folded string solutions of [8] in  $AdS_5 \times S^5$ . We will show that the energy of such configurations precisely matches with the one-loop corrected scaling dimension of the corresponding gauge theory operator, computed using the twisted XXX  $SU(2)$  spin-chain.

The  $\beta$  deformation can be understood rather simply as a modification of the superpotential of the  $\mathcal{N} = 4$  theory via

$$W = \text{Tr}[\Phi_1, [\Phi_2, \Phi_3]] \rightarrow \kappa \text{Tr}[e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2]. \quad (1.1)$$

This deformation yields an  $\mathcal{N} = 1$  superconformal field theory (SCFT) and preserves a  $U(1) \times U(1) \times U(1)_R$  global symmetry subgroup of the  $SO(6)_R$  symmetry of the  $\mathcal{N} = 4$  theory. These global symmetries translate to  $U(1)^3$  isometries in the dual geometry of [7], which was central to their construction of the supergravity (SUGRA) solution. We obtain classical string solutions which are pointlike in the  $AdS_5$  part of the geometry, and which carry large angular momenta  $J_1$  and  $J_2$  along two of the  $U(1)$  isometry directions. They are the  $\beta$ -deformed analogs (with  $\beta \ll 1$ ) of the folded string solutions of [8], and while being “folded” solutions they also wind around one of the  $U(1)$  isometry directions. These solutions are dual to gauge theory operators of the type  $\text{Tr}[\Phi_1^{J_1}\Phi_2^{J_2}] + \text{permutations}$ , with  $J_{1,2} \gg 1$ . Such operators are non-BPS operators and thus their scaling dimensions can and do get quantum corrections. Importantly, the range of  $\beta$  which we specify below and consequently the class of states that we consider, are actually distinct from those studied recently in [9, 10].

Thus the operators we are looking at are far from BMN operators [11] as they have large numbers of “impurities”. We make use of the remarkable observation made in [12] which revealed that the field theory problem of calculating the one loop anomalous dimensions of large operators such as the ones we are considering, can be reduced to the problem of diagonalising certain spin chain Hamiltonians using the well-known Bethe Ansatz. In the context of the  $\mathcal{N} = 4$  theory, striking agreements between semiclassical spinning string energies in  $AdS_5 \times S^5$  and anomalous dimensions of operators have been demonstrated in [8], [13]-[18]. The beautiful underlying integrable structures which underpin the entire “spin chain/spinning string” correspondence have been revealed in a series of papers [19]-[27]. Moreover, the idea of integrabilities has also been applied in the study of certain defect theory [28]. Finally, there are also interesting recent attempts to semiclassically quantize the entire string spectrum derived from certain subsectors of the string sigma model [29].

Extending this remarkable spin-chain/spinning string correspondence to the less supersymmetric setup of  $\mathcal{N} = 1$ ,  $\beta$  deformed theories could be of phenomenological interest, but more immediately, it might also tell us whether the above integrable structures are merely characteristics of the maximally supersymmetric theory or a more general feature of gauge/string dualities, similar to the integrability in the correspondence between two dimensional string theory and matrix quantum mechanics.

The key point that makes the  $\beta$  deformed theories special in this context is that they are obtained by a continuous, *exactly* marginal deformation of the maximally supersymmetric theory. The resultant field theory is superconformal for all values of the gauge

coupling parameter and importantly, exists at weak gauge couplings which allows perturbative calculations. The construction of the dual SUGRA background by [7] has made it possible to further explore properties of the string dual of the  $\beta$  deformation [9, 10]. (The techniques of [7] have also recently been applied to generate the deformations of  $\mathcal{N} = 1, 2$  theories and there are also many interesting related results [30]).

The integrability of the dilatation operator in the  $\beta$  deformed theory was first discussed in [31], where the one-loop anomalous dimension matrix for real  $\beta$  was found to be the “twisted” XXX spin chain Hamiltonian. We will use the  $SU(2)$  version of this Hamiltonian to compute the anomalous dimensions of the operators  $\text{Tr}[\Phi_1^{J_1}\Phi_2^{J_2}]$  for  $\beta \ll 1$ . Let us now be more specific about the range of  $\beta$  and the class of states we are studying.

One of our primary motivations is to understand the states in the theory with small  $\beta \ll 1$ . This question is particularly interesting for the theories with  $\beta = 1/n$ , where  $n$  is an integer and  $n \gg 1$ . As we discuss in detail in section 3, this  $\mathcal{N} = 1$  theory has an infinite tower of chiral primary operators of the type  $\text{Tr}[\Phi_1^{np}\Phi_2^{nq}]$  with  $(p, q)$  being nonnegative integers. Importantly, these operators are not chiral primaries in the  $\mathcal{N} = 4$  theory. However, it is fairly clear what happens to them as  $\beta \rightarrow 0$  or  $n \rightarrow \infty$ ; they get infinitely large energies or scaling dimensions. In fact all the operators of the type  $\text{Tr}[\Phi_1^{J_1}\Phi_2^{J_2}]$  in  $\mathcal{N} = 4$  theory can be accommodated within the tower of operators between the identity and  $\text{Tr}[\Phi_1^n\Phi_2^n]$  as  $n \rightarrow \infty$ . The operators of interest to us will satisfy the condition

$$\beta J_{1,2} \ll 1 \tag{1.2}$$

which, for  $\beta = 1/n$  simply means that  $J_{1,2} \ll n$  and so these operators are actually closer to the bottom of the tower, and their anomalous dimensions will get small  $\beta$ -dependent corrections. Correspondingly, in the limit of large  $J_{1,2}$ , the dual classical string states will be small  $\beta$ -deformations of the two spin folded string solution of [8]. To be precise we are taking the  $n \rightarrow \infty$  limit first and subsequently the limit  $J_{1,2} \rightarrow \infty$ . The authors of [9] considered operators for which  $\beta J_{1,2}$  was kept fixed in the large  $J_{1,2}$  limit. While it is not *a priori* clear that the two limits describe an overlapping set of states, our results indicate that they are indeed compatible. In fact, the agreement that we will find is implied by the general matching shown in [9] between the string sigma model and the continuum limit of the coherent state action for the spin-chain.

In this article we calculate the energy for the  $\beta$  deformed folded string solutions and present the transcendental equations which define the parameters classifying them. We then perform an explicit twisted spin chain analysis in the thermodynamical limit for this system, which should be regarded as a generalization of the symmetrical two cuts solutions associated with the undeformed theory. For the spin-chain/spinning string correspondence to go through, we demonstrate how the elliptic moduli in the twisted spin chain and the parameters in the folded strings should be naturally identified. Moreover, we calculated the anomalous dimension using the twisted spin chain, and showed how this can be matched with the energy of the folded strings after some elliptic modular transformations and the identification of the moduli stated earlier.

In the section 2, we shall briefly describe the exactly marginal deformations of  $\mathcal{N} = 4$  SYM, and explain how the corresponding dual geometry was obtained. In section 3, we shall present a discussion on the class of operators we are interested in. Section 4 will be devoted to the folded string solution in the Lunin-Maldacena background. The general two-cut twisted spin-chain analysis is presented in section 5. We will conclude in section 6. In appendices A and B we list some useful identities for elliptic functions and present the calculational details for the energy of the semiclassical strings.

## 2. The $\beta$ deformation of $\mathcal{N} = 4$ SYM and its dual background

The  $\mathcal{N} = 4$  SYM theory with  $SU(N)$  gauge group in four dimensions appears as the low energy world volume theory for a stack of  $N$  coincident D3-branes. In the  $\mathcal{N} = 1$  language, this theory contains one vector multiplet and three adjoint chiral multiplets  $\Phi_i$ ,  $i = 1, 2, 3$  and a superpotential of the form

$$W = \text{Tr}(\Phi_1[\Phi_2, \Phi_3]). \tag{2.1}$$

One should note that the theory, when written in the language of  $\mathcal{N} = 1$  SUSY, manifestly displays only an  $SU(3) \times U(1)_R$  subgroup of the full  $SU(4)$  R-symmetry of  $\mathcal{N} = 4$  SYM. The complexified gauge coupling of the theory  $\tau = \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi}$ , is an exactly marginal coupling parametrising a family of four dimensional field theories with sixteen supercharges. The  $SL(2, \mathbb{Z})$  electric-magnetic duality group acts on  $\tau$  and relates different theories on this fixed line.

In addition to the gauge coupling  $\tau$ , the  $\mathcal{N} = 4$  theory possesses two other  $\mathcal{N} = 1$  SUSY preserving exactly marginal deformations namely,  $\mathcal{O}_1 = h_1 \text{Tr}(\Phi_1\{\Phi_2, \Phi_3\})$  and  $\mathcal{O}_2 = h_2 \text{Tr}(\Phi_1^3 + \Phi_2^3 + \Phi_3^3)$ . Arguments for their exact marginality were given by Leigh and Strassler [5], and they have been extensively studied by earlier authors as well [6]. The two marginal couplings  $h_1$  and  $h_2$ , along with the gauge coupling  $\tau$  now parametrise a three (complex) parameter family of  $\mathcal{N} = 1$  superconformal field theories.

In this article we restrict attention to a subset of these, the so-called  $\beta$  deformations obtained by setting  $h_2$  to zero so that up to a field rescaling the superpotential of the resultant  $\mathcal{N} = 1$  theory can be expressed as

$$W = \kappa \text{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2). \tag{2.2}$$

Both  $\kappa$  and  $\beta$  will be complex in general. However, in this paper we will only be interested in the case of real  $\beta$ . We can recover the  $\mathcal{N} = 4$  theory by setting  $\beta = 0$  and  $\kappa = 1$ . The  $\beta$  deformation preserves a  $U(1)^3$  global symmetry generated by the Cartan subalgebra of the original  $SU(4)$  R-symmetry of  $\mathcal{N} = 4$  SYM. Put differently, the  $SU(3) \times U(1)_R \subset SU(4)_R$  global symmetry of  $\mathcal{N} = 4$  SYM which is manifest in the  $\mathcal{N} = 1$  language, is broken by the  $\beta$  deformation which preserves a  $U(1) \times U(1) \times U(1)_R \subset SU(3) \times U(1)_R$  global symmetry. It is also worth noting that the  $SL(2, \mathbb{Z})$  invariance of  $\mathcal{N} = 4$  SYM extends to the  $\beta$  deformed theory [40] via the following action on the couplings

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \beta \rightarrow \frac{\beta}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{2.3}$$

Various aspects of the  $\beta$  deformed theories have been extensively studied in [32]–[42].

The  $\beta$  deformed family of superconformal theories with  $U(N)$  gauge group are also known to have string theory/supergravity duals at large  $N$  and  $g_{YM}^2 N \gg 1$ . For suitably small  $\beta$  the gravity dual can be thought of as a smooth deformation of  $AdS_5 \times S^5$  and the fact that the deformation is marginal implies that it should only result in a deformation of the  $S^5$ , breaking the  $SO(6)_R$  global symmetry to  $U(1) \times U(1) \times U(1)_R$  while preserving the  $AdS_5$  metric. In [37] it was shown to lowest order in  $\beta$  that the deformation corresponds to switching on a source for the complexified IIB three form field strength  $G_3 = F_3 - \tau_s H_3$  in the  $S^5$  of  $AdS_5 \times S^5$  and that this flux back reacts and smoothly deforms the  $S^5$ .

The authors of [7] however found the complete SUGRA background dual to the field theory. In fact they demonstrated how to obtain exact SUGRA solutions for deformations that preserve a  $U(1) \times U(1)$  global symmetry. In particular the geometry has two isometries associated to the two  $U(1)$  global symmetries and thus contains a two torus. As the global symmetry of the  $\beta$  deformed theory is a subset of the  $\mathcal{N} = 4$  R-symmetry group, this two torus is contained in the  $S^5$  of the original  $AdS_5 \times S^5$  background and survives the deformation. At the classical level, the  $SL(2, \mathbb{R})$  isometry of the torus allows us to generate the new ( $\beta$ -deformed) supergravity background by considering its action on the parameter

$$\tau_{tor} = B_{12} + i\sqrt{\det[g]}, \tag{2.4}$$

where  $B_{12}$  is the component of NS two form along the torus and  $\sqrt{\det[g]}$  is the volume of the torus. Note that  $\tau_{tor}$  is distinct from the complexified string/gauge theory coupling constant  $\tau$ . Note also that  $B_{12}$  is absent in the  $AdS_5 \times S^5$  background we started with, and the relevant element of  $SL(2, \mathbb{R})$  which yields the  $\beta$  deformation acts on  $\tau_{tor}$  as

$$\tau_{tor} \rightarrow \frac{\tau_{tor}}{1 + \beta \tau_{tor}}. \tag{2.5}$$

In the newly generated background, we now have a non-vanishing, non-constant  $B_{12}$  turned on. Alternatively, we can decompose the action of (2.5) and interpret it as performing a T-duality along one of the circles of the torus, shifting the coordinate, followed by another T-duality transformation (TsT transformation [10]). After applying these steps to the original  $AdS_5 \times S^5$  with radius of curvature  $R$ , the dual supergravity solution for the  $\beta$  deformed theory in the string frame is

$$\begin{aligned} ds_{str}^2 &= R_{AdS}^2 \left[ ds_{AdS_5}^2 + \sum_{i=1}^3 (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \hat{\beta}^2 G\mu_1^2 \mu_2^2 \mu_3^2 \left( \sum_{i=1}^3 d\phi_i \right)^2 \right], \\ G &= \frac{1}{1 + \hat{\beta}^2 (\mu_1^2 \mu_2^2 + \mu_2^2 \mu_3^2 + \mu_1^2 \mu_3^2)}, \quad \hat{\beta} = R_{AdS}^2 \beta, \quad \sum_{i=1}^3 \mu_i^2 = 1, \\ R_{AdS}^4 &= 4\pi e^{\phi_0} N, \quad e^{2\phi} = e^{2\phi_0} G, \\ B^{NS} &= \hat{\beta} R_{AdS}^2 G (\mu_1^2 \mu_2^2 d\phi_1 d\phi_2 + \mu_2^2 \mu_3^2 d\phi_2 d\phi_3 + \mu_1^2 \mu_3^2 d\phi_1 d\phi_3), \\ C_2 &= -R_{AdS}^2 e^{-\phi_0} \hat{\beta} \omega_1 d\psi, \quad d\omega_1 = 12 \cos \alpha \sin^3 \alpha \sin \theta \cos \theta d\alpha d\theta, \\ F_5 &= dC_4 = 4R_{AdS}^4 e^{-\phi_0} (\omega_{AdS_5} + \omega_{S^5}). \end{aligned} \tag{2.6}$$

The  $AdS$  radius  $R_{AdS}$  is measured in units of the string scale  $\sqrt{\alpha'}$ . Here  $\omega_{AdS_5}$  and  $\omega_{S^5}$  are the volume forms for  $AdS_5$  and  $S^5$  respectively,  $\phi$  and  $\phi_0$  are the dilaton and its

expectation value,  $C_2$  and  $C_4$  are the RR two-form and four-form.  $\mu_1, \mu_2$  provide the standard parametrization of  $S^2$ :

$$\mu_1 = \sin \alpha \cos \theta, \quad \mu_2 = \sin \alpha \sin \theta \quad \mu_3 = \cos \alpha. \quad (2.7)$$

The angular variable  $\psi$  is a combination of the three toroidal directions  $\psi = \frac{1}{3}(\phi_1 + \phi_2 + \phi_3)$ , and it is related to the  $U(1)_R$  generator of the  $\mathcal{N} = 1$   $SU(2, 2|1)$  superconformal group. We can see that both  $B^{NS}$  and  $C_2$  are non-constant, therefore the associated non-trivial field strength should deform the original  $S^5$ . Topologically, the compact manifold of the solution is still  $S^5$ , and we can continuously deform it back into  $S_5$  by decreasing  $\beta$ , hence switching off the deformation parameters. We expect that for the macroscopic semiclassical strings moving in  $S^5$ , we can also continuously deform them into their counterparts in the new background. Solutions of this kind will be the focus of this paper.

The classical supergravity solution (2.6) can only be valid if the string length is much smaller than the typical size of the torus, therefore we have to supplement the usual condition  $R_{AdS} \gg 1$  with  $R_{AdS} \beta \ll 1$ .

### 3. Operators in $\beta$ deformed theory

In  $\mathcal{N} = 4$  SYM, physical states correspond to local gauge invariant operators which transform in unitary representations of the superconformal group  $SU(2, 2|4)$ , specified by the Dynkin labels  $[\Delta, s_1, s_2, r_1, r_2, r_3]$  of its bosonic subgroup  $SO(1, 1) \times SO(1, 3) \times SU(4)_R$ . Here  $\Delta$  is the scaling dimension (the eigenvalue of the dilatation operator),  $s_1$  and  $s_2$  are the two Lorentz spins of  $SO(1, 3)$ , whereas  $r_1, r_2$  and  $r_3$  give the Dynkin labels of  $SU(4)_R$ . Chiral primary operators or  $\frac{1}{2}$ -BPS states of the  $\mathcal{N} = 4$  theory are those whose scaling dimensions  $\Delta$  are uniquely determined by their R-symmetry representations. The lowest (bosonic) components of these multiplets transform in a representation of weight  $[0, k, 0]$  of  $SU(4)_R$ .

Generic gauge invariant, single trace operators correspond to closed string states in the Type IIB theory on  $AdS_5 \times S^5$ . In particular, it is well-known that single trace gauge theory operators with large R-charges have a dual description on the string sigma model side in terms of semiclassical string states [11, 43, 44] carrying large angular momentum on the  $S^5$  of  $AdS_5 \times S^5$ . We will only be interested in those states which move on the compact manifold, appearing as point particles in  $AdS_5$  and for which  $s_1$  and  $s_2$  can be set to zero.

Our focus will specifically be on semiclassical string solutions in the  $\beta$  deformed theory, corresponding to a class of operators with  $[r_1, r_2, r_3] = [J_2, J_1 - J_2, J_2]$ , *i.e.* the single trace operators of the form  $\text{Tr}(\Phi_1^{J_1} \Phi_2^{J_2}) + \text{Permutations}$ . The labels here are the charges under the unbroken  $U(1)^3 \subset SU(4)_R$  global symmetry in the presence of the  $\beta$  deformation. For generic  $\beta$ , operators with charges  $[0, k, 0]$  (or  $[k, 0, 0]$  and  $[0, 0, k]$ ) are chiral primary operators [41, 42] just as they would have been in the  $\mathcal{N} = 4$  theory.

Both  $J_1$  and  $J_2$  will be taken to be large so that unlike typical BMN operators [11], they will have very high density of impurities and we can no longer treat these states as small deviations from BPS operators. While the bare scaling dimension is equal to  $J_1 + J_2$ , it receives quantum corrections at all orders in the 't Hooft coupling  $\lambda = g_{YM}^2 N$ .

In particular, the problem of finding the one loop anomalous dimension boils down to diagonalizing a complicated mixing matrix between single trace operators formed from various inequivalent permutations of a string of  $J_1$   $\Phi_1$ 's and  $J_2$   $\Phi_2$ 's. For the  $\mathcal{N} = 4$  theory ( $\beta = 0$ ) this problem was explicitly solved in [14] using the Bethe ansatz, and was later shown to precisely match the string sigma model prediction in [8, 15].

We can generally express the scaling dimension for this class of operators in terms of the parameters in the theory as  $\Delta = \Delta[J, \frac{\lambda}{J^2}, \frac{J}{\lambda}]$ , where  $J = J_1 + J_2$ . In the presence of the  $\beta$  deformation, one expects the scaling dimensions of this class of operators to receive further  $\beta$  dependent corrections. These corrections would also appear in the energy of the corresponding semiclassical string states moving in the dual SUGRA background. The aim of this paper is to show the matching between the anomalous dimension of the gauge theory operators above and the semiclassical string energy in the  $\beta$  deformed theory, for a specific range of values of  $\beta$ .

Let us now specify the range of  $\beta$  that we are interested in. From the field theory perspective it is particularly interesting to consider theories with  $\beta = \frac{1}{n}$ , where  $n$  is a positive integer. It is known that for  $\beta = 1/n$  and  $n$  a finite integer, the  $U(N)$  field theory is in fact the low energy world volume description of a stack of  $N$  coincident D3 branes placed at the singularity of the orbifold  $\mathbb{C}^3/(\mathbb{Z}_n \times \mathbb{Z}_n)$  with discrete torsion [45, 46]. In the large  $N$  limit, keeping  $n$  small, we can take a near horizon limit to obtain the gravity dual description, namely IIB string theory propagating in  $AdS_5 \times S^5/(\mathbb{Z}_n \times \mathbb{Z}_n)$  [41, 42]. In [42] the conditions for the single trace operators to be  $\frac{1}{2}$  BPS in this theory are specified. Consider a generic single trace operator labelled  $(J_1, J_2, J_3)$

$$\text{Tr}(\Phi_1^{J_1} \Phi_2^{J_2} \Phi_3^{J_3}). \tag{3.1}$$

This operator is  $\frac{1}{2}$  BPS if

$$(J_1, J_2, J_3) = (k, 0, 0), \quad (0, k, 0), \quad (0, 0, k) \quad k \in \mathbb{N} \tag{3.2}$$

just as it would be in the  $\mathcal{N} = 4$  theory. Remarkably however, the  $\mathcal{N} = 1$  superconformal  $\beta$  deformation with  $\beta = 1/n$  has an infinite set of additional chiral primary operators labelled by

$$(J_1, J_2, J_3) = (k_1, k_2, k_3) \quad k_i = 0 \text{ mod}(n). \tag{3.3}$$

Eq. (3.2) is simply inherited from the  $\mathcal{N} = 4$  classification while we can understand Eq. (3.3) by noticing that using the F-term constraints, this class of operators actually lies in the centre of the algebra of the  $\mathcal{N} = 1$  holomorphic operators and hence commutes with the superpotential. As a result the scaling dimensions of these operators do not get quantum corrections.

By this classification, a generic two spin operator of the kind we are interested in,  $(J_1, J_2, 0)$ , would fall between two  $\frac{1}{2}$  BPS operators labelled by  $(np, nq, 0)$  and  $(n(p+1), n(q+1), 0)$ , where  $p, q$  are non-negative integers. It is interesting to ask what happens to this infinite tower of  $1/2$  BPS operators when  $n \rightarrow \infty$ , since in this limit we should recover the  $\mathcal{N} = 4$  theory which has only the BPS states (3.2). When  $n \rightarrow \infty$ , the orbifold description can no longer be valid, as the cone constructed from the  $\mathbb{C}^3/(\mathbb{Z}_n \times \mathbb{Z}_n)$  orbifold



now shrinks to a cylinder of size smaller than the string length [7, 47]. In fact in this limit  $\beta = 1/n \ll 1$  it is appropriate to describe theory in terms of the smooth SUGRA geometry (2.6) wherein the states with labels  $(np, nq, 0)$  with  $p, q \neq 0$  will have very large masses and will decouple.

In this paper we will be interested in states of the  $\beta$  deformed theory which can be understood as small deformations of the two spin states studied in [8] namely the folded closed string solution. These non-BPS operators lie in between  $(0, 0, 0)$  and  $(n, n, 0)$  states of the theory with  $\beta = 1/n \ll 1$ , and we can effectively fit all two spin  $\mathcal{N} = 1$  holomorphic operators in that range. Specifically, we are interested in the limit  $\beta J_1, \beta J_2 \ll 1$  with  $J_1, J_2 \gg 1$ . Since  $\beta = 1/n$ , these operators are actually lying close to the bottom of the tower of states between  $(0, 0, 0)$  and  $(n, n, 0)$  and can be thought of as the  $\beta$  deformed versions of the corresponding  $\mathcal{N} = 4$  states. We should point out that this limit is consistent with the requirement  $R_{AdS} \beta \ll 1$  for the supergravity solution (2.6) to be valid and enables us to find the semiclassical string solution.

Note also that the limit we are considering is different to that studied in [9] where  $\beta J_{1,2}$  was kept fixed in the limit of large  $J_1$  and  $J_2$ . The latter limit will, in principle, allow the study of all states including those that live in the middle of the abovementioned tower of states in the  $\beta$  deformed theory with  $\beta = 1/n$ .

#### 4. The folded closed string in Lunin-Maldacena background

In this section we will describe certain semiclassical solutions which are the  $\beta$  deformed versions of the folded closed string solutions of [8].

##### 4.1 General equations of motion

Let us first use the dual supergravity solution for the  $\beta$  deformed theory in (2.6) to write down the bosonic part of string sigma model Lagrangian

$$\begin{aligned}
 S_{\text{bosonic}} = & -\frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \left\{ G_{mn}^{(AdS_5)} \partial_a X^m \partial^a X^n \right. \\
 & + \sum_i (\partial_a \mu_i \partial^a \mu_i + G \mu_i^2 \partial_a \phi_i \partial^a \phi_i) + \hat{\beta}^2 G \mu_1^2 \mu_2^2 \mu_3^2 \left( \sum_i \partial_a \phi_i \right) \left( \sum_j \partial^a \phi_j \right) \\
 & \left. + 2\hat{\beta} G \epsilon^{ab} (\mu_1^2 \mu_2^2 \partial_a \phi_1 \partial_b \phi_2 + \mu_2^2 \mu_3^2 \partial_a \phi_2 \partial_b \phi_3 + \mu_1^2 \mu_3^2 \partial_a \phi_1 \partial_b \phi_3) \right\}. \tag{4.1}
 \end{aligned}$$

where  $\sqrt{\lambda} = R_{AdS}^2 / \alpha'$ . As we are interested in a perturbation of the two spin solution obtained in the  $\mathcal{N} = 4$  case [8] which only moves in the compact manifold, we shall ignore the  $AdS_5$  part for now. Treating the spacetime embedding as worldsheet fields, we can deduce the following equations of motion for the angular coordinates  $\theta$  and  $\alpha$ :

$$2\partial_a \partial^a \alpha = \sin 2\alpha \partial_a \theta \partial^a \theta + \sum_i \frac{\partial}{\partial \alpha} (G \mu_i^2) \partial_a \phi_i \partial^a \phi_i$$

$$\begin{aligned}
 & + \hat{\beta}^2 \frac{\partial}{\partial \alpha} (G \mu_1^2 \mu_2^2 \mu_3^2) \left( \sum_i \partial_a \phi_i \right) \left( \sum_i \partial^a \phi_i \right) + 2 \epsilon^{ab} \hat{\beta} \quad (4.2) \\
 & \times \left\{ \frac{\partial}{\partial \alpha} (G \mu_1^2 \mu_2^2) \partial_a \phi_1 \partial_b \phi_2 + \frac{\partial}{\partial \alpha} (G \mu_2^2 \mu_3^2) \partial_a \phi_2 \partial_b \phi_3 + \frac{\partial}{\partial \alpha} (G \mu_1^2 \mu_3^2) \partial_a \phi_1 \partial_b \phi_3 \right\}, \\
 & 2 \sin^2 \alpha \partial_a \partial^a \theta + 2 \sin 2\alpha \partial_a \alpha \partial^a \theta = \\
 & \sum_i \frac{\partial}{\partial \theta} \left( G \mu_i^2 \right) \partial_a \phi_i \partial^a \phi_i + \hat{\beta}^2 \frac{\partial}{\partial \theta} (G \mu_1^2 \mu_2^2 \mu_3^2) \left( \sum_i \partial_a \phi_i \right) \left( \sum_i \partial^a \phi_i \right) + 2 \epsilon^{ab} \hat{\beta} \\
 & \times \left\{ \frac{\partial}{\partial \theta} (G \mu_1^2 \mu_2^2) \partial_a \phi_1 \partial_b \phi_2 + \frac{\partial}{\partial \theta} (G \mu_2^2 \mu_3^2) \partial_a \phi_2 \partial_b \phi_3 + \frac{\partial}{\partial \theta} (G \mu_1^2 \mu_3^2) \partial_a \phi_1 \partial_b \phi_3 \right\}, \quad (4.3)
 \end{aligned}$$

The equations of motion for  $\phi_i$  which parametrise the three  $U(1)$  isometries of this background, yield the following conservation laws for the worldsheet densities  $\mathcal{J}^a$

$$\partial_a \mathcal{J}_1^a = \partial_a \mathcal{J}_2^a = \partial_a \mathcal{J}_3^a = 0, \quad (4.4)$$

$$\mathcal{J}_1^a = G \mu_1^2 \left\{ \partial^a \phi_1 + \hat{\beta} \epsilon^{ab} (\mu_2^2 \partial_b \phi_2 - \mu_3^2 \partial_b \phi_3) + \hat{\beta}^2 \mu_1^2 \mu_2^2 \mu_3^2 \left( \sum_i \partial^a \phi_i \right) \right\}, \quad (4.5)$$

$$\mathcal{J}_2^a = G \mu_2^2 \left\{ \partial^a \phi_2 + \hat{\beta} \epsilon^{ab} (\mu_3^2 \partial_b \phi_3 - \mu_1^2 \partial_b \phi_1) + \hat{\beta}^2 \mu_1^2 \mu_2^2 \mu_3^2 \left( \sum_i \partial^a \phi_i \right) \right\}, \quad (4.6)$$

$$\mathcal{J}_3^a = G \mu_3^2 \left\{ \partial^a \phi_3 + \hat{\beta} \epsilon^{ab} (\mu_1^2 \partial_b \phi_1 - \mu_2^2 \partial_b \phi_2) + \hat{\beta}^2 \mu_1^2 \mu_2^2 \mu_3^2 \left( \sum_i \partial^a \phi_i \right) \right\}. \quad (4.7)$$

These conservation laws arise simply because the action only depends on derivatives of  $\phi_i$  and translations in these directions are isometries.

## 4.2 Ansatz

Here we are only interested in the two spin solutions which are dual to the operator  $\text{Tr}(\Phi_1^{J_1} \Phi_2^{J_2})$ . Therefore we first restrict the semiclassical solution to be along the axis  $\alpha = \frac{\pi}{2}$  and we can see that this is consistent with the equation of motion for  $\alpha$  (4.2) and effectively sets  $\mathcal{J}_3^a$  zero. Clearly, from the  $Z_3$  discrete symmetry between  $\Phi_1, \Phi_2$  and  $\Phi_3$ , we can obtain the two spin solutions dual to the operators  $\text{Tr}(\Phi_1^{J_1} \Phi_3^{J_3})$  or  $\text{Tr}(\Phi_2^{J_2} \Phi_3^{J_3})$  by choosing alternate axes,  $\theta = 0$  or  $\theta = \frac{\pi}{2}$  respectively. Furthermore, we shall make following rotating string ansatz for the spacetime embedding

$$t = \kappa \tau, \quad \theta \equiv \theta(\sigma), \quad \phi_{1,2} = \omega_{1,2} \tau + h_{1,2}(\sigma), \quad \phi_3 = 0. \quad (4.8)$$

Here  $(\tau, \sigma)$  are the string worldsheet coordinates as usual, and the string is spinning along the  $\phi_1$  and  $\phi_2$  directions while also having a nontrivial spatial extent along  $\theta, \phi_1$  and  $\phi_2$  coordinates.

Within this ansatz the worldsheet angular momentum densities  $\mathcal{J}_1^\tau$  and  $\mathcal{J}_2^\tau$  become independent of time and only involve  $\sigma$ -dependent terms. This, along with the conservation laws  $\partial_a \mathcal{J}_{1,2}^a = 0$ , in turn implies that the densities  $\mathcal{J}_{1,2}^\sigma$  are constant and thus uniformly

distributed along the string,

$$\mathcal{J}_1^\sigma = G\mu_1^2 \left( \frac{dh_1}{d\sigma} + \hat{\beta}\mu_2^2\omega_2 \right) = C_1, \quad \mathcal{J}_2^\sigma = G\mu_2^2 \left( \frac{dh_2}{d\sigma} - \hat{\beta}\mu_1^2\omega_1 \right) = C_2, \quad (4.9)$$

$$\text{with } \mu_1 = \cos \theta, \quad \mu_2 = \sin \theta, \quad G = \frac{1}{1 + \hat{\beta}^2 \cos^2 \theta \sin^2 \theta}.$$

Here  $C_1$  and  $C_2$  are the integration constants. We will see subsequently that turning on non-zero values for  $C_1$  and  $C_2$  is necessary to incorporate the effect of the deformation parameter  $\beta$  into the solution. From (4.9), the equations of motion for  $h_1$  and  $h_2$  can be written in terms of  $\theta(\sigma)$

$$\frac{dh_1}{d\sigma} = \frac{C_1}{\mu_1^2} + \hat{\beta} \Omega_2 \mu_2^2, \quad \frac{dh_2}{d\sigma} = \frac{C_2}{\mu_2^2} - \hat{\beta} \Omega_1 \mu_1^2, \quad (4.10)$$

$$\Omega_2 = -(\omega_2 - \hat{\beta}C_1), \quad \Omega_1 = -(\omega_1 + \hat{\beta}C_2) \quad (4.11)$$

The isometries in the  $\phi_{1,2}$  directions lead to a conservation of the corresponding angular momenta  $J_1$  and  $J_2$ . These are naturally obtained as worldsheet spatial integrals over the associated charge densities  $\mathcal{J}_{1,2}^\tau$

$$J_1 = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \{ \Omega_1 \cos^2 \theta(\sigma) \}, \quad J_2 = \frac{\sqrt{\lambda}}{2\pi} \int_0^{2\pi} d\sigma \{ \Omega_2 \sin^2 \theta(\sigma) \}. \quad (4.12)$$

Note that these have exactly the same form as the expressions encountered in [8] for the two spin folded closed string in  $AdS_5 \times S^5$  with the replacement  $\omega_{1,2} \rightarrow -\Omega_{1,2}$ . Combining (4.9) and (4.10), the equation of motion for  $\theta(\sigma)$  becomes

$$\frac{d^2\theta}{d\sigma^2} = -\Omega_{12}^2 \cos \theta \sin \theta + C_2^2 \frac{\cos \theta}{\sin^3 \theta} - C_1^2 \frac{\sin \theta}{\cos^3 \theta}, \quad \Omega_{12}^2 = \Omega_2^2 - \Omega_1^2. \quad (4.13)$$

With non-zero constants  $C_1$  and  $C_2$  this equation describes a deformation of the folded closed string of [8]. Clearly, we could also choose  $\Omega_1 = \Omega_2$  and  $C_1 = C_2 = 0$  resulting in a solution with fixed  $\theta$  which is the so-called ‘‘circular’’ string. More generally, integrating (4.13) and introducing appropriate integration constants we get

$$\left( \frac{d\theta}{d\sigma} \right)^2 = \Omega_{12}^2 (\sin^2 \theta_0 - \sin^2 \theta) + C_1^2 \left( \frac{1}{\cos^2 \theta_0} - \frac{1}{\cos^2 \theta} \right) + C_2^2 \left( \frac{1}{\sin^2 \theta_0} - \frac{1}{\sin^2 \theta} \right). \quad (4.14)$$

This equation now governs the entire dynamics of our new solution, and in conjunction with (4.10) yields the following form for the Virasoro constraints

$$\kappa^2 = \Omega_1^2 \cos^2 \theta_0 + \Omega_2^2 \sin^2 \theta_0 + \left( \frac{C_1^2}{\cos^2 \theta_0} + \frac{C_2^2}{\sin^2 \theta_0} \right), \quad (4.15)$$

$$\omega_1 C_1 + \omega_2 C_2 = -(\Omega_1 C_1 + \Omega_2 C_2) = 0. \quad (4.16)$$

The first Virasoro constraint gives the energy of the string configuration

$$E^2 = \lambda \kappa^2. \quad (4.17)$$

The first two terms in (4.15) together constitute the energy of the original two spin folded string, while the last term can be thought of as a correction due to non-zero  $\beta$  in the new background. We will see below that the second Virasoro constraint (4.16) actually implies a relation between the two integration constants  $C_1$  and  $C_2$ . Equations (4.14) and (4.15) have also appeared in the two spin solutions for the so-called “Neumann-Rosochatius” integrable system discussed in [22] as a generalization of the semiclassical strings in the undeformed  $AdS_5 \times S^5$ .

### 4.3 Solutions in elliptic parametrization

Let us now solve (4.14) in the elliptic parametrization. We first introduce a new variable  $x = \sin^2 \theta$  so that the equation can then be rewritten as

$$\left(\frac{dx}{d\sigma}\right)^2 = 4\Omega_{12}^2(x_+ - x)(x_0 - x)(x - x_-), \quad (4.18)$$

where  $x_0 = \sin^2 \theta_0$  and

$$x_{\pm} = \quad (4.19)$$

$$\frac{1}{2} \left[ 1 + \left( \frac{C_1^2}{(1-x_0)\Omega_{12}^2} + \frac{C_2^2}{x_0\Omega_{12}^2} \right) \pm \sqrt{\left( 1 + \frac{C_1^2}{(1-x_0)\Omega_{12}^2} + \frac{C_2^2}{x_0\Omega_{12}^2} \right)^2 - 4\frac{C_2^2}{x_0\Omega_{12}^2}} \right]$$

or alternatively

$$\left(\frac{C_1}{\Omega_{12}}\right)^2 = (x_+ - 1)(1 - x_0)(1 - x_-), \quad \left(\frac{C_2}{\Omega_{21}}\right)^2 = x_+ x_0 x_-. \quad (4.20)$$

Using the periodicity of the closed string fields in the  $\sigma$  coordinate, we can integrate both sides of (4.18) and express the results as complete elliptic integrals (see appendix A for our conventions)

$$2\pi = \int_0^{2\pi} d\sigma = \frac{2}{\Omega_{12}} \int_{x_-}^{x_0} \frac{dx}{\sqrt{(x_+ - x)(x_0 - x)(x - x_-)}} \quad (4.21)$$

leading to

$$\Omega_{12} = \frac{2}{\pi} \frac{K(k)}{\sqrt{x_+ - x_-}}, \quad k = \frac{x_0 - x_-}{x_+ - x_-}. \quad (4.22)$$

Similarly, the angular momentum integrals of motion are

$$\begin{aligned} J_1 &= \sqrt{\lambda} \frac{\Omega_1}{2\Omega_{12}} \int_{x_-}^{x_0} \frac{dx(1-x)}{\sqrt{(x_+ - x)(x_0 - x)(x - x_-)}} \\ &= \Omega_1 \left( 1 - \left( x_+ - (x_+ - x_-) \frac{E(k)}{K(k)} \right) \right) \end{aligned} \quad (4.23)$$

$$\begin{aligned} J_2 &= \sqrt{\lambda} \frac{\Omega_2}{2\Omega_{12}} \int_{x_-}^{x_0} \frac{dx x}{\sqrt{(x_+ - x)(x_0 - x)(x - x_-)}} \\ &= \Omega_2 \left( x_+ - (x_+ - x_-) \frac{E(k)}{K(k)} \right). \end{aligned} \quad (4.24)$$

Since the variables  $k, x_0, x_+$  and  $x_-$  are related via (4.22), only three of these parameters are independent. We shall use  $k$  and  $x_0$  interchangeably wherever necessary, to simplify the expressions. Using  $\Omega_{12}^2 = \Omega_2^2 - \Omega_1^2$  and the equations above to eliminate the variables  $\Omega_1$  and  $\Omega_2$ , we can derive a transcendental equation relating the various elliptic parameters and the angular momentum quantum numbers

$$\frac{4\lambda}{\pi^2(x_+ - x_-)} = \frac{J_2^2}{(x_+K(k) - (x_+ - x_-)E(k))^2} - \frac{J_1^2}{((1 - x_+)K(k) + (x_+ - x_-)E(k))^2}. \quad (4.25)$$

In the limit of vanishing  $C_1$  and  $C_2$ , this relation reduces to the corresponding equation for the original two spin folded string in [8]. For non-zero  $C_1$  and  $C_2$  the situation is somewhat complicated: Firstly, the second Virasoro constraint (4.16) can no longer be trivially satisfied and takes the form

$$\frac{J_1 C_1}{((1 - x_+)K(k) + (x_+ - x_-)E(k))} + \frac{J_2 C_2}{(x_+K(k) - (x_+ - x_-)E(k))} = 0. \quad (4.26)$$

The semiclassical string solution of the type considered here can only exist if the integration constants  $C_{1,2}$  and the angular momenta  $J_{1,2}$  satisfy the above condition.

Moreover, the dual supergravity solution for the deformed theory with  $\beta = 1/n \ll 1$ , given by (2.6), is a smooth background. Unlike the in [45, 46] there is no orbifold action and the closed strings can only have zero or integer winding numbers. Inspecting the form of  $dh_i/d\sigma$  in (4.10) and integrating with respect to  $\sigma$ , the winding numbers depend on the combinations  $\beta J_1$  and  $\beta J_2$  which are non-integers in general, and in particular for us  $\beta J_{1,2} \ll 1$ . The terms proportional to  $C_1$  and  $C_2$  then become necessary to yield integer winding numbers. Explicitly, we obtain

$$\begin{aligned} 2\pi N_1 &= \phi_1(2\pi) - \phi_1(0) = \int_0^{2\pi} d\sigma \frac{dh_1}{d\sigma} = \int_0^{2\pi} d\sigma \left( \frac{C_1}{\mu_1^2} + \hat{\beta} \Omega_2 \mu_2^2 \right) \\ &= \frac{4C_1}{\Omega_{21}} \frac{\Pi(\alpha_1^2, k)}{(1 - x_-)\sqrt{x_+ - x_-}} + 2\pi\beta J_2, \end{aligned} \quad (4.27)$$

and

$$\begin{aligned} 2\pi N_2 &= \phi_2(2\pi) - \phi_2(0) = \int_0^{2\pi} d\sigma \frac{dh_2}{d\sigma} = \int_0^{2\pi} d\sigma \left( \frac{C_2}{\mu_2^2} - \hat{\beta} \Omega_1 \mu_1^2 \right) \\ &= \frac{4C_2}{\Omega_{21}} \frac{\Pi(\alpha_2^2, k)}{x_- \sqrt{x_+ - x_-}} - 2\pi\beta J_1, \end{aligned} \quad (4.28)$$

where

$$\alpha_1^2 = \frac{(x_0 - x_-)}{(1 - x_-)}, \quad \alpha_2^2 = -\frac{(x_0 - x_-)}{x_-}. \quad (4.29)$$

and  $\Pi(\alpha^2, k)$  is a complete elliptic integral of the third kind.

Here  $N_1$  and  $N_2$  are the integer winding numbers along the  $\phi_1$  and  $\phi_2$  circles. As we are interested in the limit where  $\beta J_1, \beta J_2 \ll 1$  we expect  $C_1$  and  $C_2$  to be parametrically

small quantities and importantly, to give a parametrically small correction to the energy expression (4.15).

Interestingly (4.14), (4.15), and (4.16) are also the equations of motion and Virasoro constraints for a folded semiclassical string configuration with arbitrary integer winding numbers propagating in the original  $AdS_5 \times S^5$  background [23]. Here we derived them purely from the new dual background as a general two spin solution for the  $\beta$  deformed theory. This observation is consistent with the relationship between semiclassical strings in the undeformed and  $\beta$  deformed backgrounds, pointed out in [10].

As noted earlier and from (4.15) we expect that  $C_1$  and  $C_2$  should be small and both should vanish as  $\beta \rightarrow 0$  as our solution should only be a small perturbation away from the original folded string solution. Naively therefore, from our discussion above, it would seem natural to set  $N_1$  and  $N_2$  to zero for the configuration corresponding to the perturbation of the two spin solution. However, it turns out that the  $\beta \rightarrow 0$  limit is somewhat subtle. In fact we find that in order for a well defined  $\beta \rightarrow 0$  limit (and  $C_1, C_2 \rightarrow 0$ ) to exist we must have  $N_2 = 1$ . This arises in the  $\beta, C_{1,2} \rightarrow 0$  limit due to a cancellation between a vanishing denominator in  $1/x_-$  and a zero of the elliptic function  $\Pi(\alpha_2^2, k)$ .

#### 4.4 Energy of states

One can in principle derive a double expansion in  $J^{-1}$  and  $\beta J \ll 1$ , ( $J = J_1 + J_2$ ) for the energy of the semiclassical string solution with string  $\alpha'$  corrections suppressed by higher powers of  $1/J$ . In such a scheme, the coefficient of each power of  $1/J$  would be a nontrivial function of  $\beta$ . In particular, the energy  $E = J + \frac{\lambda}{J} \varepsilon_1 + O(\frac{1}{J^2})$  where  $\varepsilon_1$  would be a function of  $\beta J$ . We would need to extract  $\varepsilon_1$  in order to compare with the field theoretic spin chain analysis below which yields the anomalous dimension at one loop order in the 't Hooft coupling  $\lambda$ . In order to do this systematically we would first need to expand all the elliptic modular parameters of the solutions above in powers of  $1/J$ ,

$$k = k^{(0)} + \frac{k^{(2)}}{\mathcal{J}^2} \dots, \quad x_0 = x_0^{(0)} + \frac{x_0^{(2)}}{\mathcal{J}^2} \dots, \quad x_+ = x_+^{(0)} + \frac{x_+^{(2)}}{\mathcal{J}^2} \dots, \quad x_- = x_-^{(0)} + \frac{x_-^{(2)}}{\mathcal{J}^2} \dots, \tag{4.30}$$

where  $\mathcal{J} = J/\sqrt{\lambda}$  and the expansion coefficients  $k^{(0)}, k^{(2)}, x_0^{(0)}, x_0^{(2)}, x_+^{(0)}, x_+^{(2)}, x_-^{(0)}, x_-^{(2)}$  are functions of  $\beta J$ . Substituting these into (4.20), (4.22) and (4.23), after lengthly but straightforward expansions, one can in principle rewrite  $k^{(2)}, x_+^{(2)}, x_-^{(2)}$  in terms of  $k^{(0)}, x_+^{(0)}, x_-^{(0)}$ , and derive an expression of  $\varepsilon_1$  involving only  $k^{(0)}, x_+^{(0)}, x_-^{(0)}$ . We present an alternate, simpler derivation for  $\varepsilon_1$  explicitly in the appendix B and the result is given by

$$\varepsilon_1 = \frac{2K(k^{(0)}) (E(k^{(0)}) - (1 - k^{(0)}) K(k^{(0)}))}{\pi^2} + \frac{2K(k^{(0)})^2 (x_+^{(0)} + x_-^{(0)} - 1)}{\pi^2 (x_+^{(0)} - x_-^{(0)})} \tag{4.31}$$

In the limit of vanishing  $\beta$  wherein  $x_+^{(0)} \rightarrow 1$ ,  $x_-^{(0)} \rightarrow 0$ , and  $k^{(0)} \rightarrow x_0^{(0)}$ , the expression for  $\varepsilon_1$  reduces to Eq. (2.7) in [15]. The remarkable simplicity of the expression (4.31) and the fact that in the  $\beta \rightarrow 0$  limit it matches the energy of the original folded string in  $AdS_5 \times S^5$ , together indicate that the  $\beta$  dependent correction to the semiclassical string energy in the deformed background can indeed be regarded as a small perturbation.

The  $\beta$  dependences of  $k^{(0)}$ ,  $x_+^{(0)}$  and  $x_-^{(0)}$  are given implicitly by the following transcendental equations

$$\alpha = \frac{J_2}{J} = x_+^{(0)} - (x_+^{(0)} - x_-^{(0)}) \frac{E(k^{(0)})}{K(k^{(0)})}, \quad (4.32)$$

$$C_1^{(0)} + C_2^{(0)} = 0 \longrightarrow (x_+^{(0)} - 1) (1 - x_0^{(0)}) (1 - x_-^{(0)}) = x_+^{(0)} x_0^{(0)} x_-^{(0)}, \quad (4.33)$$

$$2\pi\beta J(1 - 2\alpha) =$$

$$4 \sqrt{\frac{(x_+^{(0)} - 1) (1 - x_0^{(0)})}{(1 - x_-^{(0)}) (x_+^{(0)} - x_-^{(0)})}} \Pi(\alpha_1^2, k^{(0)}) + 4 \sqrt{\frac{x_+^{(0)} x_0^{(0)}}{x_-^{(0)} (x_+^{(0)} - x_-^{(0)})}} \Pi(\alpha_2^2, k^{(0)}) - 2\pi. \quad (4.34)$$

where  $\alpha = J_2/J$  is the filling ratio. It is from these equations that the parameters  $k^{(0)}$ ,  $x_+^{(0)}$  and  $x_-^{(0)}$  acquire dependences on  $\beta$ . One can, in principle, solve these complicated simultaneous equations by expanding in powers of  $\beta J$  from a given undeformed folded string solution. We will adopt a more direct and less messy approach.

As we will demonstrate in the next section, from results at lowest order in the  $1/J$  expansion the parameters  $k^{(0)}$ ,  $x_+^{(0)}$  and  $x_-^{(0)}$  above can be related to the moduli of the elliptic curve for the corresponding spin chain with the help of some suitable elliptic modular transformations. Following this identification of the moduli, we are then able to reproduce the precise expression for the one loop anomalous dimension  $\varepsilon_1$  above from the spin-chain analysis, provided we impose an extra condition, which happily turns out to be the Virasoro constraint for the spinning string solution.

## 5. The Twisted Spin-Chain analysis

### 5.1 The Bethe ansatz

In this section, we carry out the spin-chain analysis for the one loop anomalous dimension of the operators  $\text{Tr}[\Phi_1^{J_1} \Phi_2^{J_2}]$  in the  $\beta$  deformed theory. As established in [9, 31], the effect of turning on a real valued  $\beta$  deformation of the  $\mathcal{N} = 4$  SYM Lagrangian is to introduce a non-trivial “twisting” parameter in the original XXX  $SU(2)$  spin chain which computes the one loop anomalous dimensions of the single trace gauge invariant operators. Explicitly, in the presence of the twisting the modified Bethe equations are

$$e^{-2i\pi\beta J} \left( \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \right)^J = \prod_{j \neq k=1}^{J_2} \frac{u_k - u_j + i}{u_k - u_j - i}, \quad (5.1)$$

and

$$e^{-2i\pi\beta J_2} \prod_{j=1}^{J_2} \frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}} = 1. \quad (5.2)$$

The equation 5.2 corresponds to the modification of the trace condition due to  $\beta$ . One can perhaps most easily observe this by using the Coulomb branch condition and setting  $\beta$  to a rational number. This gives a linear shift in the the total lattice momentum of the spin chain. As we are interested in the so-called “thermodynamic” limit where  $J, J_2 \rightarrow \infty$  while keeping the filling ratio  $\alpha = \frac{J_2}{J}$  fixed, all the Bethe roots  $u_k$  are of order  $J$ . Hence we can rescale the roots,  $u_k = Jx_k$  and in terms of the rescaled Bethe roots the expression for the energy takes the form

$$\gamma = \frac{\lambda}{8\pi^2 J^2} \sum_{k=1}^{J_2} \frac{1}{x_k^2}. \tag{5.3}$$

The twisted thermodynamical Bethe equation gets modified into

$$\frac{1}{x_k} = 2\pi(\beta J - n_k) + \frac{2}{J} \sum_{j \neq k=1}^{J_2} \frac{1}{(x_k - x_j)}. \tag{5.4}$$

Summing over the index  $k$  on both sides of the Bethe equation yields the condition

$$\sum_{k=1}^{J_2} \frac{1}{x_k} = 2\pi \left( (\beta J) J_2 - \sum_{k=1}^{J_2} n_k \right). \tag{5.5}$$

We first recall that in the absence of  $\beta$  the spin chain configuration for the folded string does not have any lattice momentum. Then, comparing equation (5.5) with the logarithm of (5.2) in the thermodynamic limit, we deduce

$$\sum_{k=1}^{J_2} n_k = 0. \tag{5.6}$$

Moreover, since we are looking for the lowest energy distribution of the Bethe roots, from the energy (5.3) we infer that for  $J_2$  an even number, the  $n_k$  should come in pairs taking the values  $\pm 1$ . This means that to a first approximation, the Bethe roots condense into two cuts labeled by  $\mathcal{C}_\pm$  spread around  $n_k = \pm 1$  respectively. (While the condition  $\sum_{k=1}^{J_2} n_k = 0$  can also be satisfied by choosing other set of  $n_k$ , these would correspond to the multi-cut distributions and would have higher energy  $\gamma$ .)

In the undeformed theory this was essentially the complete story - the two cuts were symmetrical about the imaginary axis with the Bethe roots repelling one another via Coulomb-like forces on each cut. In the undeformed case the entire distribution could be described by the end-points of either one of the two cuts, as the branch cuts are simply mirror images of each other.

Turning on a  $\beta \neq 0$  gives different shifts to the roots on the  $\mathcal{C}_+$  and  $\mathcal{C}_-$  cuts and as a result, breaks the symmetry about the imaginary axis. This asymmetry is clearly exhibited in the Bethe equation (5.4). The four end-points of the branch cuts must now be treated independently, although their locations are only small perturbations around the symmetric  $\beta = 0$  case. This then requires us to perform a general two-cut analysis. The appearance of general two-cut distributions is not surprising, given the appearance of elliptic functions describing the semiclassical string solutions in the previous section. The Riemann surface associated to a two-cut distribution is a torus on which elliptic functions are naturally defined as analytic functions.



## 5.2 The two-cut Riemann surface

Let us now solve for this two cut distribution by first labelling the four end points as  $x_1 > x_2 > x_3 > x_4$ . The two cuts are given by  $\mathcal{C}^+ \in [x_1, x_2]$  and  $\mathcal{C}^- \in [x_3, x_4]$ . Our analysis will follow the one in [26] for the untwisted spin chain.

Define the density and the resolvent of the Bethe roots in the usual way

$$\rho(y) = \frac{1}{J} \sum_{k=1}^{J_2} \delta(y - x_k), \quad G(x) = \frac{1}{J} \sum_{k=1}^{J_2} \frac{1}{x - x_k} = \int_{\mathcal{C}} \frac{dy \rho(y)}{x - y}, \quad (5.7)$$

where  $\mathcal{C} = \mathcal{C}_+ \cup \mathcal{C}_-$ . The resolvent  $G(x)$  is an analytic function with branch cuts at  $\mathcal{C}^\pm$  and its discontinuity across a cut gives the density of Bethe roots.

The filling fraction of the Bethe roots on each cut can be defined as

$$\mathcal{S}_\pm = \int_{\mathcal{C}_\pm} dx \rho(x) = \frac{1}{2\pi i} \oint_{\mathcal{C}_\pm} dx G(x), \quad \mathcal{S}_+ + \mathcal{S}_- = \alpha, \quad (5.8)$$

with  $\alpha$  given by  $J_2/J$ . The resolvent function  $G(x)$  allows us to rewrite the the momentum and energy conditions elegantly as contour integrals

$$2\pi\alpha(\beta J) = \int_{\mathcal{C}} dx \frac{\rho(x)}{x} = \frac{i}{2\pi} \oint_{\mathcal{C}} dx \frac{G(x)}{x}, \quad (5.9)$$

$$\gamma = \frac{\lambda}{8\pi^2 J} \int_{\mathcal{C}} dx \frac{\rho(x)}{x^2} = \frac{i\lambda}{16\pi^3 J} \oint_{\mathcal{C}} dx \frac{G(x)}{x^2}. \quad (5.10)$$

The equalities above can be proven by exchanging the order of integrations and deforming the contour. Note that (5.8) also implies  $\mathcal{S}_+ = \mathcal{S}_-$  which can be proven by a contour deformation argument. In our case this is consistent with the earlier cyclicity condition  $\sum_{k=1}^{J_2} n_k = 0$ .

Following [26], we now define the so-called “quasi-momentum” to be

$$p(x) = G(x) - \frac{1}{2x} + \pi(\beta J), \quad (5.11)$$

which is a double valued function of  $x$  which satisfies the “integer condition”

$$p(x + i\epsilon) + p(x - i\epsilon) = 2\pi n_\pm, \quad x \in \mathcal{C}_\pm, \quad n_\pm = \pm 1. \quad (5.12)$$

Naively, the extra  $\pi(\beta J)$  in the definition of  $p(x)$  does not appear to affect the value of  $\gamma$  obtained from (5.10), as its contribution in the contour integral is a double pole which gives vanishing residue. However, the parameters entering  $p(x)$  namely, the locations of the branch points  $x_1, x_2, x_3$  and  $x_4$  can and do depend on  $\beta$  and hence the situation is somewhat subtle.

To proceed further, let us first describe the Riemann surface for  $p(x)$ . It consists of two sheets with branch cuts at  $\mathcal{C}_+$  and  $\mathcal{C}_-$ , *i.e.* it is a torus. Following the discussion on the quasi-momentum in [26], the additional term  $\pi(\beta J)$  does not change the fact that  $dp(x)$  is

a meromorphic differential on its Riemann surface with only double poles at  $x = 0$  on the two sheets. We can describe this Riemann surface by a hyperelliptic curve

$$\Sigma : \quad y^2 = \prod_{k=1}^4 (x - x_k) = x^4 + f_1 x^3 + f_2 x^2 + f_3 x + f_4, \quad (5.13)$$

and the meromorphic differential is given by

$$dp(x) = \frac{g(x)dx}{y}, \quad (5.14)$$

where  $g(x)$  is a rational function.

The precise form of  $g(x)$  can be fixed by asymptotic behaviour at large and small  $x$ . First consider the behaviour as  $x \rightarrow \infty$ . In this limit  $G(x) \rightarrow \alpha/x$  and therefore  $dp(x) \rightarrow (1/2 - \alpha)dx/x^2$ . This condition restricts  $g(x)$  to be  $\sum_{n \leq 1} a_n x^{n-1}$ , with  $a_1 = (\frac{1}{2} - \alpha)$ . The presence of a double pole in  $dp(x)$  at  $x = 0$  then further truncates the expansion for  $g(x)$  to  $a_1 + \frac{a_0}{x} + \frac{a_{-1}}{x^2}$ , so that near  $x = 0$

$$dp(x) = \frac{dx}{y} \left( \left( \frac{1}{2} - \alpha \right) + \frac{a_0}{x} + \frac{a_{-1}}{x^2} \right) = \frac{dx}{2x^2} + O(1). \quad (5.15)$$

Matching coefficients in the above expansion, we deduce that

$$a_{-1} = \frac{\sqrt{f_4}}{2} = \frac{\sqrt{\prod_{k=1}^4 x_k}}{2}, \quad a_0 = \frac{f_3}{4\sqrt{f_4}} = \frac{-\sqrt{\prod_{k=1}^4 x_k}}{4} \left( \sum_{k=1}^4 \frac{1}{x_k} \right). \quad (5.16)$$

Notice that for the undeformed theory where the Bethe roots condense into symmetrical cuts,  $a_0$  was identically zero. This, however, is not the case for the general two-cut solution we are dealing with here.

### 5.3 Elliptic parametrization

The quasi-momentum  $p(x)$  is necessarily a single valued function on each sheet (for our case without the condensate), namely the integral of the meromorphic differential  $dp(x)$  along a contour enclosing  $\mathcal{C}_+$  or  $\mathcal{C}_-$  vanishes [26]. This gives us the so-called ‘‘A-cycle’’ integration condition

$$\begin{aligned} 0 &= \oint_{\mathcal{A}_+} dp(x) = 2i \int_{x_2}^{x_1} dx \frac{a_1 + \frac{a_0}{x} + \frac{a_{-1}}{x^2}}{\sqrt{(x_1 - x)(x - x_2)(x - x_3)(x - x_4)}} \\ &= \frac{2i}{\sqrt{(x_1 - x_3)(x_2 - x_4)}} \left\{ \left[ (1 - 2\alpha) - \frac{(x_1 x_2 + x_3 x_4)}{2\sqrt{x_1 x_2 x_3 x_4}} \right] K(r) + \frac{(x_1 - x_3)(x_2 - x_4)}{2\sqrt{x_1 x_2 x_3 x_4}} E(r) \right\}. \end{aligned} \quad (5.17)$$

The integer condition (5.12) translates into the ‘‘B-cycle’’ integration condition

$$\begin{aligned} 4\pi &= \oint_{\mathcal{B}} dp(x) = 2 \int_{x_3}^{x_2} dx \frac{a_1 + \frac{a_0}{x} + \frac{a_{-1}}{x^2}}{\sqrt{(x_1 - x)(x_2 - x)(x - x_3)(x - x_4)}} \\ &= \frac{2}{\sqrt{(x_1 - x_3)(x_2 - x_4)}} \\ &\quad \times \left\{ \left[ (1 - 2\alpha) - \frac{(x_1 x_4 + x_2 x_3)}{2\sqrt{x_1 x_2 x_3 x_4}} \right] K(r') - \frac{(x_1 - x_3)(x_2 - x_4)}{2\sqrt{x_1 x_2 x_3 x_4}} E(r') \right\}. \end{aligned} \quad (5.18)$$

The elliptic modulus  $r$  and its complement  $r' = 1 - r$  are given by the cross ratios

$$r = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)}, \quad r' = \frac{(x_1 - x_4)(x_2 - x_3)}{(x_1 - x_3)(x_2 - x_4)}. \quad (5.19)$$

From the A and B-cycle conditions and using the Legendre relation (A.6) we rewrite  $K(r)$  as

$$K(r) = -\frac{\sqrt{(x_1 - x_3)(x_2 - x_4)}}{8\sqrt{x_1 x_2 x_3 x_4}}. \quad (5.20)$$

We shall use this result later.

The filling ratios  $\mathcal{S}_{\pm}$  can also be explicitly computed from the contour integrals of  $p(x)$  enclosing each cut,

$$\begin{aligned} \mathcal{S}_+ &= \frac{1}{\pi} \int_{x_2}^{x_1} dy \int dx \frac{a_1 + \frac{a_0}{x} + \frac{a_{-1}}{x^2}}{\sqrt{(x_1 - x)(x - x_2)(x - x_3)(x - x_4)}} \\ &= \frac{(1 - 2\alpha)(x_1 - x_4)}{\pi \sqrt{(x_1 - x_3)(x_2 - x_4)}} \Pi(\omega^2, r) - \frac{x_2 x_3 (x_1 - x_4)}{\pi \sqrt{(x_1 - x_3)(x_2 - x_4) x_1 x_2 x_3 x_4}} \Pi(\omega_1^2, r) \\ &\quad - \frac{x_2 (x_1 - x_3)(x_1 - x_4) K(r)}{2\pi \sqrt{(x_1 - x_3)(x_2 - x_4) x_1 x_2 x_3 x_4}} \left( 1 - \frac{x_1 (x_2 - x_4) E(r)}{x_2 (x_1 - x_4) K(r)} \right) + \alpha, \end{aligned} \quad (5.21)$$

and

$$\begin{aligned} \mathcal{S}_- &= \frac{1}{\pi} \int_{x_4}^{x_3} dy \int dx \frac{a_1 + \frac{a_0}{x} + \frac{a_{-1}}{x^2}}{\sqrt{(x_1 - x)(x_2 - x)(x_3 - x)(x - x_4)}} \\ &= \frac{(1 - 2\alpha)(x_1 - x_4)}{\pi \sqrt{(x_1 - x_3)(x_2 - x_4)}} \Pi(\tilde{\omega}^2, r) - \frac{x_2 x_3 (x_1 - x_4)}{\pi \sqrt{(x_1 - x_3)(x_2 - x_4) x_1 x_2 x_3 x_4}} \Pi(\tilde{\omega}_1^2, r) \\ &\quad + \frac{x_3 (x_1 - x_4)(x_2 - x_4) K(r)}{2\pi \sqrt{(x_1 - x_3)(x_2 - x_4) x_1 x_2 x_3 x_4}} \left( 1 - \frac{x_4 (x_1 - x_3) E(r)}{x_3 (x_1 - x_4) K(r)} \right) + \alpha. \end{aligned} \quad (5.22)$$

Here the elliptic moduli  $\omega^2$ ,  $\omega_1^2$ ,  $\tilde{\omega}^2$  and  $\tilde{\omega}_1^2$  are given by

$$\omega^2 = -\frac{(x_3 - x_4)}{(x_1 - x_3)}, \quad \omega_1^2 = -\frac{x_1(x_3 - x_4)}{x_4(x_1 - x_3)}, \quad \tilde{\omega}^2 = \frac{r}{\omega^2}, \quad \tilde{\omega}_1^2 = \frac{r}{\omega_1^2}. \quad (5.23)$$

In deriving  $\mathcal{S}_+$  and  $\mathcal{S}_-$ , we have repeatedly used the identities (A.3) and (A.4) in the appendix A to simplify the expressions. The appearance of elliptic integrals of the third kind in these expressions for the filling ratio is rather suggestive since we have also encountered these in the semiclassical string computation. From the identity (A.5), we can deduce that there are in fact only three independent elliptic moduli in the expressions namely,  $r$ ,  $\omega^2$  and  $\omega_1^2$ . This is exactly the same number as in the semiclassical string analysis. It is also easily checked that that  $\mathcal{S}_+ + \mathcal{S}_- = \alpha$ .

While the condition  $\mathcal{S}_+ = \mathcal{S}_-$  can be trivially satisfied in the situation with symmetrical cuts, in the deformed case equating the expressions (5.21) and (5.22) yields a non-trivial constraint on the end points of the cuts. Although the two cuts still contain equal numbers of Bethe roots, the potential felt along each cut is now different and the roots can have different distributions. The two branch cuts are therefore not necessarily symmetrical or of equal length.

## 5.4 Matching the string and spin-chain parameters

At this point we can make further concrete connections with the semiclassical string analysis and relate the elliptic parameters in the two pictures. In this context the key equation is the vanishing A-cycle condition (5.17) which can be rewritten using the identity (A.3) as an expression for the filling ratio

$$\alpha = \frac{1}{2} - \frac{(x_1x_2 + x_3x_4)}{4\sqrt{x_1x_2x_3x_4}} + \frac{(x_1 - x_4)(x_2 - x_3)}{4\sqrt{x_1x_2x_3x_4}} \frac{\mathbb{E}\left[-\frac{r}{1-r}\right]}{\mathbb{K}\left[-\frac{r}{1-r}\right]}. \quad (5.24)$$

Comparing this with the corresponding string theory expression (4.32) we see that they are rather similar. Indeed, for the correspondence between the spin-chain and semiclassical string solutions to work, they have to be equal! This requires us to naturally identify the moduli from the two different calculations as follows:

$$\begin{aligned} x_+^{(0)} &= \frac{1}{2} - \frac{(x_1x_2 + x_3x_4)}{4\sqrt{x_1x_2x_3x_4}}, & x_-^{(0)} &= \frac{1}{2} - \frac{(x_1x_3 + x_2x_4)}{4\sqrt{x_1x_2x_3x_4}}, \\ x_0^{(0)} &= \frac{1}{2} - \frac{(x_1x_4 + x_2x_3)}{4\sqrt{x_1x_2x_3x_4}}, & k^{(0)} &= -\frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_2 - x_3)}. \end{aligned} \quad (5.25)$$

As an elementary check for these expressions, we notice that in the limit of two symmetrical cuts,  $x_4 \rightarrow -x_1$  and  $x_3 \rightarrow -x_2$ , which was the natural Bethe roots distribution to consider when  $\beta$  the deformation parameter is set to zero,

$$x_+^{(0)} = 1, \quad x_-^{(0)} = 0, \quad x_0^{(0)} = k_0^{(0)} = -\frac{(x_1 - x_2)^2}{4x_1x_2}. \quad (5.26)$$

Here we have followed the convention in [26] so that  $\sqrt{x_1x_2x_3x_4} \rightarrow -x_1x_2$  in the limit. The first two expressions above simply give the correct constants for the undeformed case, whereas the third expression coincides with the identification between the elliptic moduli in the original folded string and spin chain calculations given in [22].

Furthermore, for the semiclassical string solution in the previous section to be valid, the Virasoro constraint (4.26) has to be satisfied to all orders in  $\beta$ . Using the identification of the moduli above, we can re-express the lowest order Virasoro constraint equation as

$$\left(\sum_{i=1}^4 x_i\right) \left(\sum_{i=1}^4 \frac{1}{x_i}\right) = 0. \quad (5.27)$$

This condition imposes an extra constraint on the end points  $x_1 \dots x_4$  in order for the correspondence between the spin chain and the semiclassical string to be consistent. In the case of the undeformed symmetrical cuts distribution, the constraint is trivially satisfied. However now we need a nontrivial relation between the branch point locations which leads to either  $\sum_{i=1}^4 x_i = 0$  or  $\sum_{i=1}^4 \frac{1}{x_i} = 0$  so that the Virasoro constraints are satisfied. As we see below the latter condition  $\sum_{i=1}^4 1/x_i = 0$  is also the necessary condition for the matching of the anomalous dimension calculated from the two sides of the correspondence.

We can now calculate the anomalous dimension  $\gamma$  using the definition (5.10) after using the identity (A.5) which yields

$$\gamma = \frac{1}{32\pi^2} \left\{ \left( \frac{1}{x_2} - \frac{1}{x_4} \right) \left( \frac{1}{x_1} - \frac{1}{x_3} \right) \frac{E(r)}{K(r)} - \frac{\left( \left( \frac{1}{x_1} - \frac{1}{x_3} \right) + \left( \frac{1}{x_2} - \frac{1}{x_4} \right) \right)^2}{4} \right\}. \quad (5.28)$$

This expression, albeit simple, at first sight looks nothing like the corresponding expression for the energy  $\varepsilon_1$  we derived earlier (4.31) from the string solution. But using (5.20) for  $K(r)$  and the elliptic modular transformations (A.3) we can massage the expression above into the form

$$\begin{aligned} \gamma &= \frac{2K\left(\frac{-r}{1-r}\right) \left\{ E\left(\frac{-r}{1-r}\right) - \left(\frac{1}{r}\right) K\left(\frac{-r}{1-r}\right) \right\}}{\pi^2} - \frac{\left( \left( \frac{1}{x_1} + \frac{1}{x_4} \right) - \left( \frac{1}{x_2} + \frac{1}{x_3} \right) \right)^2}{128\pi^2} \\ &= \frac{2K(k^{(0)}) \left\{ E(k^{(0)}) - (1 - k^{(0)}) K(k^{(0)}) \right\}}{\pi^2} - \frac{\left( \left( \frac{1}{x_1} + \frac{1}{x_4} \right) - \left( \frac{1}{x_2} + \frac{1}{x_3} \right) \right)^2}{128\pi^2}. \end{aligned} \quad (5.29)$$

In the second line we have used the identification of the moduli in (5.25). Now as we can see,  $\gamma$  looks remarkably similar to  $\varepsilon_1$  (4.31) and the two would be identical if we could set the second term in  $\varepsilon_1$  and in  $\gamma$  equal to each other. Equating these two using (5.20) and (5.25) we find,

$$\frac{2K(k^{(0)})^2 (x_+^{(0)} + x_-^{(0)} - 1)}{\pi^2 (x_+^{(0)} - x_-^{(0)})} = \frac{\left( \frac{1}{x_1} + \frac{1}{x_3} \right) \left( \frac{1}{x_2} + \frac{1}{x_4} \right)}{32\pi^2}. \quad (5.30)$$

The necessary condition for the semiclassical string and spin-chain calculations to match then becomes

$$4 \left( \frac{1}{x_1} + \frac{1}{x_3} \right) \left( \frac{1}{x_2} + \frac{1}{x_4} \right) = \left( \left( \frac{1}{x_1} + \frac{1}{x_4} \right) - \left( \frac{1}{x_2} + \frac{1}{x_3} \right) \right)^2, \quad (5.31)$$

But this is precisely the condition  $\sum_{i=1}^4 \frac{1}{x_i} = 0$  which satisfies the second Virasoro constraint! This striking match between the spinning string analysis and the twisted spin chain calculation for the anomalous dimension is the main result of our paper.

## 6. Summary and future direction

In this paper, we have explicitly constructed a two spin semiclassical folded string solution in the Lunin-Maldacena background. This solution is a folded configuration with nontrivial winding. It should be regarded as the natural deformation of the two-spin semiclassical folded string solution in the original  $AdS_5 \times S^5$  under the effect of non-trivial deformation parameter  $\beta$ . We then calculated its energy to the lowest order in an expansion in powers of  $\frac{\lambda}{J^2}$ . This then gives a non-trivial  $\beta$  dependent correction to the energy of the original semiclassical strings calculated in [8].

We then performed an explicit twisted spin chain analysis for a general two-cut Bethe roots distribution. We naturally identified the  $\beta$  dependent end points of the cuts with the moduli in the elliptic functions characterizing the folded string solution. With these identifications, the anomalous dimension calculated using the twisted spin chain was shown to precisely coincide with the energy of the folded string, after we impose the appropriate Virasoro constraint. The striking match provides another non-trivial check for the (one loop) integrability in the less supersymmetric backgrounds. Effectively, we have also demonstrated the role played by the second Virasoro constraint in this analysis. (Some earlier discussions can be found in [23].).

A natural extension of this work would be trying to understand how conserved charges other than the energy can be matched between the twisted spin chain and spinning strings, as was done in [24]. It is known that the twisted spin chain is an integrable system with a large number of conserved charges. On the other hand the class of semiclassical string solutions we have in this paper belongs to the so-called Neumann-Rosochatius integrable system [23]. It would be both physically and mathematically interesting to see how the conserved charges in the twisted spin chain can also arise in this class of semiclassical solutions. Following [24], this would presumably require a generalization of the Bäcklund transformation to the semiclassical string solution with non-trivial winding number. We leave this line of investigation for future work.

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### A. Definitions and identities for the complete elliptic functions

Here we list the convention of the complete elliptic integrals used in this paper:

$$K[k] = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k \sin^2 \phi}}, \quad E[k] = \int_0^{\frac{\pi}{2}} d\phi \sqrt{1 - k \sin^2 \phi} \quad (\text{A.1})$$

$$\Pi(\omega^2, k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{(1 - \omega^2 \sin^2 \phi) \sqrt{1 - k \sin^2 \phi}}. \quad (\text{A.2})$$

In addition, some very useful identities for identifying the moduli in the sigma model and spin chain calculations are

$$K[k] = \frac{1}{\sqrt{1 - k}} K \left[ \frac{-k}{1 - k} \right], \quad E[k] = \sqrt{1 - k} E \left[ \frac{-k}{1 - k} \right], \quad (\text{A.3})$$

$$\Pi(\omega^2, k) = \frac{1}{(1 - \omega^2) \sqrt{1 - k}} \Pi \left( -\frac{\omega^2}{1 - \omega^2}, -\frac{k}{1 - k} \right), \quad (\text{A.4})$$

$$\Pi(\omega^2, k) + \Pi(k/\omega^2, k) = K[k] + \frac{\pi}{2} \sqrt{\frac{\omega^2}{(1-\omega^2)(\omega^2-k)}}. \quad (\text{A.5})$$

The Legendre relation is given by

$$E(k)K(1-k) + E(1-r)K(r) - K(r)K(1-r) = \frac{\pi}{2}. \quad (\text{A.6})$$

## B. Explicit derivation of the folded string energy

Here we describe how one can obtain the first order correction to the folded string energy  $\varepsilon_1$  in (4.30).

Starting from the Virasoro constraint (4.15), the energy  $E$ , the total angular momentum  $J$  and the  $t'$  Hooft coupling  $\lambda$  are related to  $\kappa$  and  $\mathcal{J}$  by  $E = \sqrt{\lambda}\kappa$  and  $J = \sqrt{\lambda}\mathcal{J}$ . Using the expansions (4.29), we can expand  $\kappa$  to the first order  $\frac{1}{\mathcal{J}}$

$$\kappa = \mathcal{J} + \frac{1}{\mathcal{J}} \left( \frac{(\alpha - x_0^{(0)})}{(1-\alpha)} \mathcal{F} + \frac{2K(k^{(0)})^2 (x_+^{(0)} + x_-^{(0)} - 1)}{\pi^2 (x_+^{(0)} - x_-^{(0)})} \right). \quad (\text{B.1})$$

Here we have also used the zeroth order relation (4.31) for  $\alpha$  to simplify the expression. The function  $\mathcal{F}$  is given by

$$\begin{aligned} \mathcal{F} &= \frac{x_+^{(2)}}{\alpha} \left( 1 - \frac{E(k^{(0)})}{K(k^{(0)})} \right) + \frac{x_-^{(2)}}{\alpha} \frac{E(k^{(0)})}{K(k^{(0)})} \\ &+ \frac{k^{(2)} (x_+^{(0)} - x_-^{(0)})}{2\alpha (1 - k^{(0)}) k^{(0)}} \left( (1 - k^{(0)}) \left( 1 - \frac{E(k^{(0)})}{K(k^{(0)})} \right)^2 + k^{(0)} \left( \frac{E(k^{(0)})}{K(k^{(0)})} \right)^2 \right). \end{aligned} \quad (\text{B.2})$$

To work out  $x_+^{(2)}$ ,  $x_-^{(2)}$  and  $k^{(2)}$  in terms of  $x_+^{(0)}$ ,  $x_-^{(0)}$ , and  $k^{(0)}$ , one can in principle expand (4.25), (4.26) and the sum of (4.27) and (4.28) to the first order in  $\frac{1}{\mathcal{J}}$  and solve the simultaneous equations. However, we are only interested in  $\kappa$  here, let us focus on the expansion of (4.25) at the order  $\frac{1}{\mathcal{J}}$ ,

$$\begin{aligned} \frac{-4K(k^{(0)})^2}{\pi^2 (x_+^{(0)} - x_-^{(0)})} &= \frac{2x_+^{(2)}}{\alpha(1-\alpha)} \left( 1 - \frac{E(k^{(0)})}{K(k^{(0)})} \right) + \frac{2x_-^{(2)}}{\alpha(1-\alpha)} \frac{E(k^{(0)})}{K(k^{(0)})} \\ &+ \frac{k^{(2)} (x_+^{(0)} - x_-^{(0)})}{\alpha(1-\alpha) (1 - k^{(0)}) k^{(0)}} \left( (1 - k^{(0)}) \left( 1 - \frac{E(k^{(0)})}{K(k^{(0)})} \right)^2 + k^{(0)} \left( \frac{E(k^{(0)})}{K(k^{(0)})} \right)^2 \right) \end{aligned} \quad (\text{B.3})$$

the expression on the right hand side is proportional to  $\mathcal{F}$ ! We can substitute the left hand side into  $\mathcal{F}$  and  $\kappa$ , hence the simple expression for  $\varepsilon_1$  in (4.30).

## References

- [1] J.M. Maldacena, *The large- $N$  limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [[hep-th/9711200](#)].

- [2] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)].
- [3] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [[hep-th/9802109](#)].
- [4] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large- $N$  field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [[hep-th/9905111](#)].
- [5] R.G. Leigh and M.J. Strassler, *Exactly marginal operators and duality in four-dimensional  $N = 1$  supersymmetric gauge theory*, *Nucl. Phys.* **B 447** (1995) 95 [[hep-th/9503121](#)].
- [6] A. Parkes and P.C. West, *Finiteness in rigid supersymmetric theories*, *Phys. Lett.* **B 138** (1984) 99;  
P.C. West, *The yukawa beta function in  $N = 1$  rigid supersymmetric theories*, *Phys. Lett.* **B 137** (1984) 371;  
D.R.T. Jones and L. Mezincescu, *The chiral anomaly and a class of two loop finite supersymmetric gauge theories*, *Phys. Lett.* **B 138** (1984) 293;  
S. Hamidi, J. Patera and J.H. Schwarz, *Chiral two loop finite supersymmetric theories*, *Phys. Lett.* **B 141** (1984) 349;  
S. Hamidi and J.H. Schwarz, *A realistic finite unified theory?*, *Phys. Lett.* **B 147** (1984) 301;  
W. Lucha and H. Neufeld, *Finiteness of quantum field theories and supersymmetry*, *Phys. Lett.* **B 174** (1986) 186; *Finite quantum field theories*, *Phys. Rev.* **D 34** (1986) 1089;  
D.R.T. Jones, *Coupling constant reparametrization and finite field theories*, *Nucl. Phys.* **B 277** (1986) 153;  
A.V. Ermushev, D.I. Kazakov and O.V. Tarasov, *Finite  $N = 1$  supersymmetric grand unified theories*, *Nucl. Phys.* **B 281** (1987) 72;  
X.D. Jiang and X.J. Zhou, *A criterion for existence of finite to all orders  $N = 1$  sym theories*, *Phys. Rev.* **D 42** (1990) 2109.
- [7] O. Lunin and J. Maldacena, *Deforming field theories with  $U(1) \times U(1)$  global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [[hep-th/0502086](#)].
- [8] S. Frolov and A.A. Tseytlin, *Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors*, *Phys. Lett.* **B 570** (2003) 96 [[hep-th/0306143](#)].
- [9] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for superconformal deformations of  $N = 4$  super Yang-Mills theory*, *JHEP* **07** (2005) 045 [[hep-th/0503192](#)].
- [10] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, *JHEP* **05** (2005) 069 [[hep-th/0503201](#)].
- [11] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from  $N = 4$  super Yang-Mills*, *JHEP* **04** (2002) 013 [[hep-th/0202021](#)].
- [12] J.A. Minahan and K. Zarembo, *The Bethe-Ansatz for  $N = 4$  super Yang-Mills*, *JHEP* **03** (2003) 013 [[hep-th/0212208](#)].
- [13] S. Frolov and A.A. Tseytlin, *Multi-spin string solutions in  $AdS_5 \times S^5$* , *Nucl. Phys.* **B 668** (2003) 77 [[hep-th/0304255](#)].
- [14] N. Beisert, J.A. Minahan, M. Staudacher and K. Zarembo, *Stringing spins and spinning strings*, *JHEP* **09** (2003) 010 [[hep-th/0306139](#)].
- [15] N. Beisert, S. Frolov, M. Staudacher and A.A. Tseytlin, *Precision spectroscopy of AdS/CFT*, *JHEP* **10** (2003) 037 [[hep-th/0308117](#)].



- [16] A.A. Tseytlin, *Spinning strings and AdS/CFT duality*, hep-th/0311139.
- [17] A.A. Tseytlin, *Semiclassical strings and AdS/CFT*, hep-th/0409296.
- [18] N. Beisert, *The dilatation operator of  $N = 4$  super Yang-Mills theory and integrability*, *Phys. Rept.* **405** (2005) 1 [hep-th/0407277].
- [19] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the  $AdS_5 \times S^5$  superstring*, *Phys. Rev. D* **69** (2004) 046002 [hep-th/0305116].
- [20] N. Beisert, *The complete one-loop dilatation operator of  $N = 4$  super Yang-Mills theory*, *Nucl. Phys. B* **676** (2004) 3 [hep-th/0307015].
- [21] N. Beisert and M. Staudacher, *The  $N = 4$  SYM integrable super spin chain*, *Nucl. Phys. B* **670** (2003) 439 [hep-th/0307042].
- [22] G. Arutyunov, S. Frolov, J. Russo and A.A. Tseytlin, *Spinning strings in  $AdS_5 \times S^5$  and integrable systems*, *Nucl. Phys. B* **671** (2003) 3 [hep-th/0307191].
- [23] G. Arutyunov, J. Russo and A.A. Tseytlin, *Spinning strings in  $AdS_5 \times S^5$ : new integrable system relations*, *Phys. Rev. D* **69** (2004) 086009 [hep-th/0311004].
- [24] G. Arutyunov and M. Staudacher, *Matching higher conserved charges for strings and spins*, *JHEP* **03** (2004) 004 [hep-th/0310182].
- [25] L.F. Alday, G. Arutyunov and A.A. Tseytlin, *On integrability of classical superstrings in  $AdS_5 \times S^5$* , *JHEP* **07** (2005) 002 [hep-th/0502240].
- [26] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [hep-th/0402207].
- [27] V.A. Kazakov and K. Zarembo, *Classical/quantum integrability in non-compact sector of AdS/CFT*, *JHEP* **10** (2004) 060 [hep-th/0410105];  
N. Beisert, V.A. Kazakov and K. Sakai, *Algebraic curve for the  $SO(6)$  sector of AdS/CFT*, hep-th/0410253;  
S. Schafer-Nameki, *The algebraic curve of 1-loop planar  $N = 4$  SYM*, *Nucl. Phys. B* **714** (2005) 3 [hep-th/0412254];  
N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *The algebraic curve of classical superstrings on  $AdS_5 \times S^5$* , hep-th/0502226; *The algebraic curve of classical superstrings on  $AdS_5 \times S^5$* , hep-th/0502226.
- [28] K. Okamura, Y. Takayama and K. Yoshida, *Open spinning strings and AdS/dCFT duality*, *JHEP* **01** (2006) 112 [hep-th/0511139].
- [29] N. Dorey and B. Vicedo, in preparation.
- [30] U. Gursoy and C. Núñez, *Dipole deformations of  $N = 1$  sym and supergravity backgrounds with  $U(1) \times U(1)$  global symmetry*, *Nucl. Phys. B* **725** (2005) 45 [hep-th/0505100];  
S. Penati, A. Santambrogio and D. Zanon, *Two-point correlators in the beta-deformed  $N = 4$  SYM at the next-to-leading order*, *JHEP* **10** (2005) 023 [hep-th/0506150];  
A. Mauri, S. Penati, A. Santambrogio and D. Zanon, *Exact results in planar  $N = 1$  superconformal Yang-Mills theory*, *JHEP* **11** (2005) 024 [hep-th/0507282];  
N.P. Bobev, H. Dimov and R.C. Rashkov, *Semiclassical strings in lunin-maldacena background*, hep-th/0506063;  
R.C. Rashkov, K.S. Viswanathan and Y. Yang,  *$Ts \dots sT$  transformation on  $AdS_5 \times S^5$  background*, hep-th/0509058.

- [31] D. Berenstein and S.A. Cherkis, *Deformations of  $N = 4$  SYM and integrable spin chain models*, *Nucl. Phys. B* **702** (2004) 49 [[hep-th/0405215](#)].
- [32] N. Dorey, *S-duality, deconstruction and confinement for a marginal deformation of  $N = 4$  SUSY Yang-Mills*, *JHEP* **08** (2004) 043 [[hep-th/0310117](#)].
- [33] N. Dorey, *A new deconstruction of little string theory*, *JHEP* **07** (2004) 016 [[hep-th/0406104](#)].
- [34] T.J. Hollowood and S. Prem Kumar, *An  $N = 1$  duality cascade from a deformation of  $N = 4$  SUSY Yang-Mills*, *JHEP* **12** (2004) 034 [[hep-th/0407029](#)].
- [35] N. Dorey and T.J. Hollowood, *On the coulomb branch of a marginal deformation of  $N = 4$  SUSY Yang-Mills*, *JHEP* **06** (2005) 036 [[hep-th/0411163](#)].
- [36] O. Aharony and S.S. Razamat, *Exactly marginal deformations of  $N = 4$  SYM and of its supersymmetric orbifold descendants*, *JHEP* **05** (2002) 029 [[hep-th/0204045](#)].
- [37] O. Aharony, B. Kol and S. Yankielowicz, *On exactly marginal deformations of  $N = 4$  SYM and type-IIB supergravity on  $AdS_5 \times S^5$* , *JHEP* **06** (2002) 039 [[hep-th/0205090](#)].
- [38] F. Benini, *The coulomb branch of the Leigh-Strassler deformation and matrix models*, *JHEP* **12** (2004) 068 [[hep-th/0411057](#)].
- [39] V. Niarchos and N. Prezas, *BMN operators for  $N = 1$  superconformal Yang-Mills theories and associated string backgrounds*, *JHEP* **06** (2003) 015 [[hep-th/0212111](#)].
- [40] N. Dorey, T.J. Hollowood and S.P. Kumar, *S-duality of the Leigh-Strassler deformation via matrix models*, *JHEP* **12** (2002) 003 [[hep-th/0210239](#)].
- [41] D. Berenstein and R.G. Leigh, *Discrete torsion,  $AdS/CFT$  and duality*, *JHEP* **01** (2000) 038 [[hep-th/0001055](#)].
- [42] D. Berenstein, V. Jejjala and R.G. Leigh, *Marginal and relevant deformations of  $N = 4$  field theories and non-commutative moduli spaces of vacua*, *Nucl. Phys. B* **589** (2000) 196 [[hep-th/0005087](#)].
- [43] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys. B* **636** (2002) 99 [[hep-th/0204051](#)].
- [44] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in  $AdS_5 \times S^5$* , *JHEP* **06** (2002) 007 [[hep-th/0204226](#)].
- [45] M.R. Douglas, *D-branes and discrete torsion*, [hep-th/9807235](#).
- [46] M.R. Douglas and B. Fiol, *D-branes and discrete torsion, II*, *JHEP* **09** (2005) 053 [[hep-th/9903031](#)].
- [47] A. Adams and M. Fabinger, *Deconstructing noncommutativity with a giant fuzzy moose*, *JHEP* **04** (2002) 006 [[hep-th/0111079](#)].